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On Some Maximum Area Problems II

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1. Introduction

In [1], we addressed the maximum area problem of quadrilaterals whose four side lengths are prescribed. A necessary and sufficient condition for a quadrilateral to achieve the maximum area was obtained. Similar result was also extended to the case of n-gons with their n side lengths prescribed. In this paper, we continue to consider other questions concerning maximum area of some geometrical figures.

2. The Maximum Area Achievable by an n-Gon with One Free Side Length

Example 1 Suppose that a triangle has two sides specified. It is interesting to find out under what conditions the triangle attains the maximal area.

Let ΔABC be a triangle with $|AB| = l_1$, $|BC| = l_2$, and $\angle ABC = \theta$ (see Figure 1). Then the area $\mathcal{A}$ of ΔABC is given by

$$\mathcal{A} = \tfrac{1}{2} l_1 l_2 \sin\theta .$$

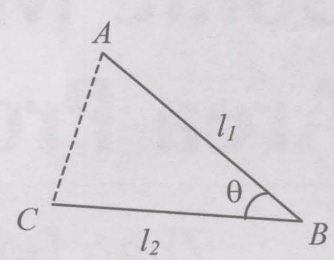

Figure 1

It is obvious that $\mathcal{A}$ attains the maximal value when $\theta = \frac{\pi}{2}$, that is, when ΔABC is a right-angled triangle.

Now, suppose that only three sides of a quadrilateral are given, we also wonder under what condition it achieves the maximal area.

Let $ABCD$ be a quadrilateral and the lengths of its three sides AB, BC, and CD be given as l_1, l_2, and l_3, respectively (see Figure 2). Join AC, and let $\angle ABC = \theta$, $\angle ADC = \psi$. Thus the area $\mathcal{A}$ of the quadrilateral $ABCD$ is given by

$$A = \frac{1}{2} l_1 l_2 \sin\theta + \frac{1}{2} l_3 \, | AD | \sin\psi . \qquad (1)$$

On Some Maximum Area Problems II

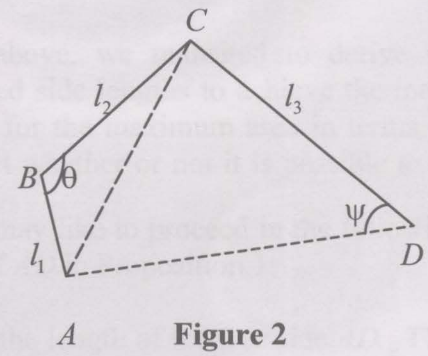

Figure 2

Let $l = |AD|$. Since AC is a function of the variable θ, l is a function of the variables θ and ψ. Hence $\mathcal{A}$ is a function of θ and ψ, and we can write $\mathcal{A} = \mathcal{A}(\theta, \psi)$.

From calculus, for a pair (θ^*, ψ^*) to optimize $\mathcal{A}$ in (1), the following conditions must be satisfied by θ^* and ψ^*:

$$\frac{\partial \mathcal{A}}{\partial \theta} = \frac{1}{2} l_1 l_2 \cos\theta + \frac{1}{2} l_3 \frac{\partial l}{\partial \theta} \sin\psi = 0, \tag{2}$$

$$\frac{\partial \mathcal{A}}{\partial \psi} = \frac{1}{2} l_3 \frac{\partial l}{\partial \psi} \sin\psi + \frac{1}{2} l_3 l \cos\psi = 0. \tag{3}$$

In view of

$$l_1^2 + l_2^2 - 2 l_1 l_2 \cos\theta = l_3^2 + l^2 - 2 l_3 l \cos\psi, \tag{4}$$

we obtain

$$\frac{\partial l}{\partial \theta} = \frac{l_1 l_2 \sin\theta}{l - l_3 \cos\psi}, \tag{5}$$

and

$$\frac{\partial l}{\partial \psi} = \frac{l_3 l \sin\psi}{l_3 \cos\psi - l} \tag{6}$$

provided that $l \neq l_3 \cos\psi$ (see Remark 1 below for the explanation why the case $l = l_3 \cos\psi$ can be excluded).

Hence, subject to $l_1 \neq 0$ and $l_2 \neq 0$, (2) and (5) yield

$$l \cos\theta^* = l_3 \cos(\theta^* + \psi^*), \tag{7}$$

On Some Maximum Area Problems II

and (3) together with (6) yield

$$l_3 = l\cos\psi^*. \tag{8}$$

Then (8) implies that $\angle ACD = \frac{\pi}{2}$.

Using (8), we obtain from (7)

$$\cos\theta^* = \cos(\theta^* + 2\psi^*), \tag{9}$$

$$\therefore\ \theta^* + 2\psi^* = 2\pi - \theta^*,$$

that is, $\theta^* + \psi^* = \pi$. (10)

We can infer from (10) that for the quadrilateral $\mathcal{A}$ to achieve the maximal area, it must be cyclic. Furthermore, since $\angle ACD = \frac{\pi}{2}$, the side AD whose length is not specified must lie on the diameter of the circle which circumscribes $\mathcal{A}$.

Remark 1: If $l - l_3\cos\psi = 0$, then $AC = l_3\sin\psi$. Let $\Delta ACD'$ be any triangle such that $CD' = l_3$, and $AD' > AD$ (see Figure 3). Let α denote $\angle ACD'$.

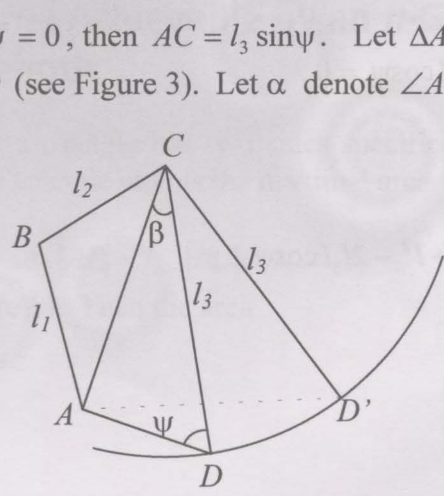

Figure 3

In considering ΔACD and $\Delta ACD'$, it is clear that $l - l_3\cos\psi = 0$ does not correspond to the maximal area that $\mathcal{A}$ can achieve because $\frac{1}{2}AC \cdot l_3\sin\beta < \frac{1}{2}AC \cdot l_3\sin\alpha$ holds for any $0 < \beta < \alpha < \pi$.
We now summarize the above results in the following proposition:

Proposition 1: A quadrilateral $ABCD$ for which the three sides AB, BC, and CD have prescribed lengths achieves the maximal area if and only if it is cyclic and the side AD is a diameter of the circumscribed circle.

On Some Maximum Area Problems II

In the above, we managed to derive the conditions for a quadrilateral with three prescribed side lengths to achieve the maximum area. We have not been able to find a formula for the maximum area in terms of the given side lengths. Actually, we do not know yet whether or not it is possible to obtain a closed form formula for it. Interested

readers may like to proceed in the following way to obtain a closed form formula for the length of AD in Proposition 1:

Let l be the length of the free side AD. The maximum area achievable by $ABCD$ is given by (see Proposition 1 of [1]):

$$\mathcal{A}(l)=\tfrac{1}{4}\sqrt{(l_1+l_2+l_3-l)(l_1+l_2+l-l_3)(l_1+l_3+l-l_2)(l_2+l_3+l-l_1)}\,. \tag{11}$$

Thus if one can find the value of l at which $\mathcal{A}(l)$ is maximum, then that value of $\mathcal{A}$ is also the maximal value achievable by the set of all quadrilaterals with three side lengths prescribed.

Exercise 1: Using elementary calculus, determine the value(s) of l for which $\mathcal{A}$ in (11) achieves its maximum value.

Nowadays, various dynamic geometry software, such as the Geometer's Sketchpad, are available for use in the teaching of mathematics in schools. Those who are familiar with any of such software may like to try the following exercise.

Exercise 2: How would you use the dynamic software to draw a quadrilateral $ABCD$ where AB, BC, and CD have prescribed lengths so that the area is a maximum?

(**Note**: It is still not known to the authors whether one can construct the quadrilateral in Exercise 2 using only rulers and compasses).

Exercise 3: Find the condition(s) for which an n-gon with (n-1) prescribed side lengths achieves maximum area.

3. Maximum Area of A Segment With Prescribed Arc Length

Next we consider a related but different type of maximum area problem. Consider all the segments in circles for which the arc lengths have prescribed value l (see Figure 4). We wish to determine that particular segment which achieves maximum area.

Consider a circle with radius r. Let θ be the angle subtended by the arc AB at the centre O of the circle. Thus the area of the segment $\mathcal{S}$ is given by

$$\mathcal{S}=\frac{\theta}{2\pi}\pi r^2-\frac{1}{2}r^2\sin\theta\,. \tag{12}$$

On Some Maximum Area Problems II

As $\theta = l/r$, so S is a function of r. Obviously r and θ satisfy $\frac{l}{2\pi} \le r$ and $0<\theta \le 2\pi$. Clearly, S achieves its maximum value neither at $r=0$ nor at $r=l$.

We shall first identify the stationary point(s) of S.

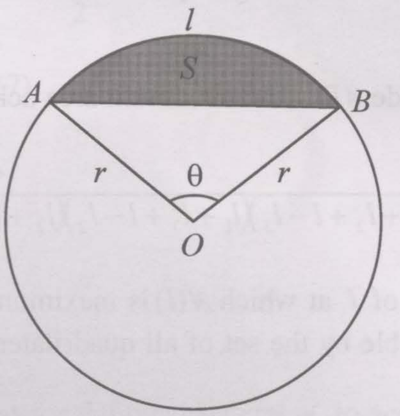

Figure 4

Notice that $\dfrac{d\theta}{dr} = -\dfrac{\theta}{r}$, hence

$$\frac{dS}{dr} = r\left[\theta - \sin\theta - \tfrac{1}{2}\theta\,(1-\cos\theta)\,\right]. \tag{13}$$

and

$$\frac{d^2 S}{dr^2} = \left[\,\theta - \sin\theta - \tfrac{1}{2}\theta\,(1-\cos\theta)\,\right] + \tfrac{1}{2}r\left(\tfrac{d\theta}{dr}\right)\left[(1-\cos\theta) - \theta\sin\theta\,\right]$$

$$= \left[\,\theta - \sin\theta - \tfrac{1}{2}\theta\,(1-\cos\theta)\,\right] - \tfrac{1}{2}\theta\left[(1-\cos\theta) - \theta\sin\theta\,\right]. \tag{14}$$

If r^* is a stationary point, and $\theta^* = {}^{l}\!/_{r^*}$, then ${}^{dS}\!/_{dr}\,|_{r^*} = 0$,

that is,

$$\theta^*(1+\cos\theta^*) = 2\sin\theta^*, \tag{15}$$

or

$$\cos\tfrac{\theta^*}{2}\left[\tfrac{\theta^*}{2}\cos\tfrac{\theta^*}{2} - \sin\tfrac{\theta^*}{2}\right] = 0. \tag{16}$$

To solve equation (17), we consider the function $f(x) = x\cos x - \sin x$. Since $f'(x) = -x\sin x < 0$ holds for all $x \in (0,\pi)$, hence the unique stationary point of S is $\theta^* = \pi$, that is $r^* = {}^{l}\!/_{\pi}$.

On Some Maximum Area Problems II

Moreover

$$\left.\frac{d^2 S}{dr^2}\right|_{\theta=\theta^*} = -\frac{\pi}{2}(1-\cos\pi) < 0.$$

In summary, the function S has a unique stationary point in the interval $\frac{l}{2\pi} \le r < \infty$, and this stationary point is a global maximum point. Hence S achieves the absolute maximum value at this point. In this case, the radius of the circle is $r^* = l/\pi$, and the area of the segment is $\frac{l^2}{2\pi}$.

4. Discussion

In this paper, we examined two problems concerning maximum area of some plane geometrical figures. In each case, we look for the maximum area achievable by a type of plane figures with part of their side/arc lengths prescribed. Interested readers may consider similar problems for other type of figures, such as pentagons, hexagons, or n-gons.

References

[1] Zhao, D.S., Yap, S.F., and Lee, C.S., On some maximum area problems I, *Mathematical Medley*, Vol **29**, No 2. (2002), 78-85.